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THE APPROXIMATE FIXED POINT PROPERTY IN BANACH AND HYPERBOLIC SPACES

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ABSTRACT
A geometrical characterization is given for those convex subsets of a Banach space (more generally a hyperbolic space) which possess the approximate fixed point property for nonexpansive mappings.

1. Introduction

Let C be a closed convex subset of a Banach space X. C is said to have the approximate fixed point property for nonexpansive mappings (AFPP) if for every nonexpansive mapping $T: C \mapsto C$, i.e.

$$|Tx - Ty| \le |x - y| \qquad \text{for all } x, y \in C,$$

$\inf\{\,|x - Tx|\,\big|\, x \in C\} = 0$. It is natural to look for a geometrical characterization of those C which possess the AFPP. The best result to date is that of Reich [7]: a closed convex subset of a reflexive Banach space has the AFPP if and only if it is linearly bounded (i.e. its intersection with any line is bounded). As was noted in [7], this result is not valid in all Banach spaces. Actually, we shall see in Section 3 that this characterization is valid for a Banach space X if and only if X is reflexive.

In this note we shall present a more general geometric characterization of the AFPP that is valid in an arbitrary Banach space. In fact, our result is true even for a more general class of metric spaces with a convexity structure, namely hyperbolic spaces. In Section 2 we shall give a definition of hyperbolic spaces and prove our main theorem on the characterization of the AFPP in those spaces. In Section 3

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we restrict our attention to the Banach space case. The problem we are interested in is whether in every Banach space there is an unbounded closed convex subset which has the AFPP. We have been unable to solve this problem for all Banach spaces, but we shall see that the only case that still needs to be resolved is of an isomorphic image of l_1.

2. The AFPP in hyperbolic spaces

Let (X,ρ) be a complete metric space. Assume that there is a family M of metric lines (isometric images of R) in X such that for every $x,y \in X$, $x \neq y$, there is a unique metric line in M that passes through x and y. The closed metric segment connecting x and y will be denoted by $[x,y]$. For every $0 \leq t \leq 1$ we shall denote by $(1-t)x \oplus ty$ the unique point $z \in [x,y]$ satisfying $\rho(x,z) = t\rho(x,y)$ and $\rho(z,y) = (1-t)\rho(x,y)$.

We shall say that X, or more precisely (X,ρ,M), is a hyperbolic space if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho(y,z)$$

for all x, y and z in X.

An equivalent requirement is that

$$\rho((1-t)x \oplus tz, (1-t)y \oplus tw) \leq (1-t)\rho(x,y) + t\rho(z,w)$$

for all x, y, z and w in X and all $0 \leq t \leq 1$.

Hyperbolic spaces were introduced by Kirk [4] and were studied in [8]. We mention briefly some examples of these spaces. Clearly all Banach spaces are also hyperbolic spaces. So is the Hilbert ball B equipped with the hyperbolic metric [3, p. 104] and B^n, the Cartesian product of n Hilbert balls, equipped with the hyperbolic metric (see [6]). Another example, the details of which will appear elsewhere, is the open unit ball of $L(H)$, the space of all bounded linear self-mappings of a complex Hilbert space H, with the hyperbolic metric. Finite dimensional examples, which are less interesting for our purpose here, are given by Hadamard manifolds, simply connected complete Riemannian manifolds of nonpositive curvature (see [1]). New examples can be constructed from old ones by a product procedure which is described in [8]. We remark that the AFPP (and the FPP) in the Hilbert ball are discussed in [3, p. 149].

Throughout this section X will denote a hyperbolic space. A subset $C \subset X$ is said to be convex if $[x,y] \in C$ whenever $x,y \in C$. A mapping $T: C \mapsto C$ is said to be nonexpansive if $\rho(Tx,Ty) \leq \rho(x,y)$ for all $x,y \in C$. We shall say that C has the

AFPP if $\inf\{\rho(x,Tx) \mid x \in C\} = 0$ for every nonexpansive mapping $T: C \mapsto C$. Next we introduce the concepts of directional curve and directional sequence that will be essential in the sequel.

DEFINITION 2.1. A curve $\gamma: [0,\infty) \mapsto X$ is said to be directional (with constant b) if there is $b \geq 0$ such that

$$t - s - b \leq \rho(\gamma(s), \gamma(t)) \leq t - s$$

for all $t \geq s \geq 0$.

A sequence $\{x_n\}_{n\geq 1} \subset X$ is said to be directional if:

(1) $\rho(x_1, x_n) \to \infty$ as $n \to \infty$,

(2) there is $b \geq 0$ such that

$$\rho(x_{n_1}, x_{n_l}) \geq \sum_{i=1}^{l-1} \rho(x_{n_i}, x_{n_{i+1}}) - b$$

for all $n_1 < n_2 < \cdots < n_l$.

DEFINITION 2.2. A convex subset C is called directionally bounded if it contains no directional curves.

LEMMA 2.3. *A convex subset is directionally bounded if and only if it contains no directional sequences.*

PROOF. Suppose C contains a directional curve $\gamma(t)$ with a constant b. Choose any positive sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \uparrow \infty$ and define

$$x_i = \gamma(t_i), \qquad i \geq 1.$$

For $n_1 < n_2 < \cdots < n_l$ we have:

$$\begin{aligned} \rho(x_{n_l}, x_{n_1}) &= \rho(\gamma(t_{n_l}), \gamma(t_{n_1})) \geq t_{n_l} - t_{n_1} - b \\ &= \sum_{i=1}^{l-1} (t_{n_{i+1}} - t_{n_i}) - b \geq \sum_{i=1}^{l-1} \rho(x_{n_i}, x_{n_{i+1}}) - b. \end{aligned}$$

Conversely, if C contains a directional sequence $\{x_n\}_{n=1}^{\infty}$ with constant b we define

$$t_1 = 0, \qquad t_n = \sum_{i=1}^{n-1} \rho(x_i, x_{i+1})$$

for $n \geq 2$, and $\gamma(t_n) = x_n$ for $n \geq 1$. We extend γ to all of R^+ by $\gamma(t) = (1 - a_t)x_n \oplus a_t x_{n+1}$ where $t_n \leq t < t_{n+1}$ and $a_t = (t - t_n)/\rho(x_n, x_{n+1})$.

If $t_{n+1} > t \geq t_n \geq t_{m+1} > s \geq t_m$, then

$$\begin{aligned}\rho(\gamma(t),\gamma(s)) &\geq \rho(\gamma(t_{n+1}),\gamma(s)) - \rho(\gamma(t_{n+1}),\gamma(t))\\ &\geq \rho(\gamma(t_{n+1}),\gamma(t_m)) - \rho(\gamma(s),\gamma(t_m)) - \rho(\gamma(t_{n+1}),\gamma(t))\\ &= \rho(x_{n+1},x_m) - (s - t_m) - (t_{n+1} - t)\\ &\geq \sum_{i=m}^{n} \rho(x_i,x_{i+1}) - b - (s - t_m) - (t_{n+1} - t)\\ &= t - s - b.\end{aligned}$$

We can now state our main theorem:

THEOREM 2.4. *A convex subset $C \subset X$ has the AFPP if and only if it is directionally bounded.*

PROOF OF NECESSITY. Suppose C contains a directional curve $\gamma(t)$ with a constant b. We define $T: C \mapsto C$ by $Tx = \gamma(A_x + 1 + b)$ where $A_x \equiv \rho(\gamma(0),x)$. It is easy to see that T is nonexpansive. In addition, for each $x \in C$ we have

$$\begin{aligned}\rho(Tx,x) &= \rho(\gamma(A_x + 1 + b),x)\\ &\geq \rho(\gamma(A_x + 1 + b),\gamma(0)) - A_x\\ &\geq A_x + 1 + b - b - A_x = 1\end{aligned}$$

hence C does not have the AFPP.

Note the similarity of the above construction to the one given in [5]. In order to prove the sufficiency part of the theorem we need some preliminary results.

Let C be a closed convex subset of X and $T: C \mapsto C$ a nonexpansive mapping. For any $x \in C$ and $t > 0$ consider the mapping $S: C \mapsto C$ defined by

$$Sy = \frac{1}{1+t} x \oplus \frac{t}{1+t} Ty.$$

S is a strict contraction, hence by Banach's fixed point theorem, it has a unique fixed point in C which we shall denote by $J_t x$. The mappings $\{J_t\}_{t>0}$ thus defined are easily seen to be nonexpansive and are called the resolvents of T, just as in Banach spaces.

The resolvent identity

$$J_t x = J_s\left(\frac{s}{t} x \oplus \left(1 - \frac{s}{t}\right) J_t x\right)$$

for any $x \in C$ and $0 < s \leq t$ can be easily verified. The next lemma appears in [8] in a more general setting, but we include a proof for completeness.

For u and v in X and $s > 0$ we shall denote by $(1+s)u \ominus sv$ the unique point w on the metric line connecting u and v that satisfies $\rho(w, u) = s\rho(u, v)$ and $\rho(w, v) = (1+s)\rho(u, v)$.

LEMMA 2.5. $\forall x \in C$,

$$\lim_{t\to\infty} \rho(x, J_t x)/t = \inf_{y\in C} \rho(y, Ty).$$

PROOF. By the resolvent identity we have for $t \geq s > 0$,

$$\rho(x, J_s x) \geq \rho(x, J_t x) - \rho(J_t x, J_s x) \geq \frac{s}{t}\,\rho(x, J_t x),$$

hence $\{\rho(x, J_t x)/t \mid t > 0\}$ is nonincreasing and

$$\lim_{t\to\infty} \rho(x, J_t x)/t = L$$

exists. Since $\rho(x, J_t x)/t = \rho(J_t x, TJ_t x)$, it is clear that $L \geq d = \inf\{\rho(y, Ty) \mid y \in C\}$.

In order to prove the reverse inequality we fix $y \in C$ and $s > 0$. For $t \geq s$ we have

$$\frac{s}{t}\,x \oplus \left(1 - \frac{s}{t}\right) J_t x = (1+s) J_t x \ominus sTJ_t x$$

hence

$$\begin{aligned}
\rho(y, J_t x) &\leq \rho\left((1+s)y \ominus sTy, \frac{s}{t}\,x \oplus \left(1 - \frac{s}{t}\right) J_t x\right) \\
&\leq \frac{s}{t}\,\rho((1+s)y \ominus sTy, x) + \left(1 - \frac{s}{t}\right)\rho((1+s)y \ominus sTy, J_t x) \\
&\leq \frac{s}{t}\,\rho((1+s)y \ominus sTy, y) + \frac{s}{t}\,\rho(y, x) \\
&\quad + \left(1 - \frac{s}{t}\right)\rho((1+s)y \ominus sTy, y) + \left(1 - \frac{s}{t}\right)\rho(y, J_t x) \\
&= s\rho(y, Ty) + \frac{s}{t}\,\rho(y, x) + \left(1 - \frac{s}{t}\right)\rho(y, J_t x).
\end{aligned}$$

So we conclude that $\rho(y, J_t x)/t \le \rho(y, Ty) + \rho(y, x)/t$. Letting $t \to \infty$ we get $L \le \rho(y, Ty)$. Since y was arbitrary, $L \le d$ and the result follows.

For $x \in C$ and a positive sequence $\{t_i\}_{i=1}^\infty$ we construct a sequence $\{y_i\}_{i=1}^\infty \in C$ as follows:

$$y_1 = J_{t_1}x, \quad y_{i+1} = \frac{s_i}{s_{i+1}}\, y_i \oplus \frac{1}{s_{i+1}t_{i+1}}\, J_{t_{i+1}}x, \quad i \ge 1, \tag{2.1}$$

where

$$s_j = \sum_{k=1}^{j} \frac{1}{t_k}, \qquad j \ge 1.$$

Note that in normed spaces

$$y_j = \left(\sum_{i=1}^{j} J_{t_i}x/t_i\right)\Big/ s_j.$$

Lemma 2.6. *Let $\{y_i\}$ be defined by* (2.1). *Then, for $m \ge 1$ and $t \ge \max\{t_i \mid 1 \le i \le m\}$,*

$$\rho(y_m, J_t x) \le \left(1 - \frac{m}{s_m t}\right)\rho(x, J_t x).$$

Proof. We use induction on m. The case $m = 1$ is clear from the resolvent identity. Suppose the result is true for m. Then

$$\begin{aligned}\rho(y_{m+1}, J_t x) &\le \frac{s_m}{s_{m+1}}\,\rho(y_m, J_t x) + \frac{1}{s_{m+1}t_{m+1}}\,\rho(J_{t_{m+1}}x, J_t x)\\ &\le \frac{s_m}{s_{m+1}}\,\rho(x, J_t x) - \frac{m}{s_{m+1}t}\,\rho(x, J_t x) + \frac{1 - t_{m+1}/t}{s_{m+1}t_{m+1}}\,\rho(x, J_t x)\\ &= \rho(x, J_t x) - \frac{m+1}{s_{m+1}t}\,\rho(x, J_t x).\end{aligned}$$

Lemma 2.7. *Let $\{y_i\}_{i=1}^\infty$ be defined by* (2.1). *Then for every $m \ge 1$,*

$$\rho(y_m, x) \ge md/s_m, \qquad \textit{where } d \equiv \inf_{y \in C} \rho(y, Ty).$$

Proof. Fix any $t \ge \max\{t_i \mid 1 \le i \le m\}$. Then by Lemma 2.6,

$$\rho(y_m, x) \ge \rho(x, J_t x) - \rho(y_m, J_t x) \ge \frac{m}{s_m}\,\rho(x, J_t x)/t \ge md/s_m.$$

We are now ready to complete the proof of Theorem 2.4.

CONCLUSION OF THE PROOF OF THEOREM 2.4. If (a closed) C does not have the AFPP, then there is a nonexpansive mapping $T: C \mapsto C$ such that $\inf\{\rho(y, Ty) \mid y \in C\} = d > 0$. We shall show that C is not directionally bounded. Fix any $x \in C$. We shall construct a sequence $\{y_i\}_{i=1}^{\infty}$ defined by (2.1) with an appropriate choice of $\{t_i\}_{i=1}^{\infty}$. We choose t_1 such that

$$\rho(x, J_{t_1}x)/t_1 \le d + \frac{1}{2},$$

so $y_1 = J_{t_1}x$. Having chosen $t_1, t_2, \ldots, t_m$, and therefore $y_1, y_2, \ldots, y_m$, we next choose t_{m+1} such that

(i) $t_{m+1} \ge 2t_m$,

(ii) $\rho(J_{t_{m+1}}x, y_m)/t_{m+1} \le d + 1/2^{m+1}$,

and define y_{m+1} as in (2.1). The existence of t_{m+1} is guaranteed by Lemma 2.5. We claim that $\{y_m\}_{m=1}^{\infty}$ thus defined is a directional sequence.

It is enough to show that

$$\left\{ \sum_{i=1}^{n-1} \rho(y_i, y_{i+1}) - \rho(y_1, y_n) \right\}_{n \ge 2}$$

is bounded.

By our construction

$$\rho(y_i, y_{i+1}) = \rho(J_{t_{i+1}}x, y_i)/(s_{i+1}t_{i+1}) \le (d + 1/2^{i+1})/s_{i+1}$$

for every i, so by Lemma 2.7 we have, for $n \ge 2$,

$$\begin{aligned}
\sum_{i=1}^{n-1} \rho(y_i, y_{i+1}) - \rho(y_1, y_n) &\le \sum_{i=1}^{n-1} (d + 1/2^{i+1})/s_{i+1} - dn/s_n + t_1(d + 1/2) \\
&= \sum_{i=1}^{n} [(d + 1/2^i)/s_i - d/s_n] \le t_1 + d \sum_{i=1}^{n-1} \left(\frac{1}{s_i} - \frac{1}{s_n} \right) \\
&\le t_1 + t_1^2 d \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{t_j} \le t_1 + t_1^2 d \sum_{i=1}^{n-1} \frac{2}{t_{i+1}} \le t_1(2d + 1).
\end{aligned}$$

Hence $\{y_i\}$ is indeed directional, and by Lemma 2.3 this completes the proof.

3. The Banach space case

Throughout this section X will denote an infinite dimensional real Banach space. We are interested in finding unbounded convex subsets that are directionally bounded, hence have the AFPP by Theorem 2.4. As it turns out, in Banach spaces there is a useful criterion which enables us to check whether a convex subset is directionally bounded or not. We shall denote by $S(X)$ and $S(X^*)$ the unit spheres of X and X^* respectively. For $x \in X$ we shall denote the set

$$\{f \in X^*; |f| = |x| \text{ and } f(x) = |x|^2\}$$

by $J(x)$. We begin with a lemma:

LEMMA 3.1. *Let $\gamma(t)$ be a directional curve with constant b in a Banach space X. Then there is a functional $f \in S(X^*)$ such that*

$$t - s - b \le f(\gamma(t) - \gamma(s)) \le t - s \qquad \text{for all } 0 \le s \le t.$$

PROOF. For $r > b$ consider $f_r \in J((\gamma(r) - \gamma(0))/|\gamma(r) - \gamma(0|)$ and let f be a weak* limit of a subset of $\{f_r;\ r > b\}$ as r tends to infinity. For $t \ge s \ge 0$ take $r > \max(t, b)$. Then

$$\begin{aligned} f_r(\gamma(t) - \gamma(s)) &= f_r(\gamma(r) - \gamma(0)) - f_r(\gamma(r) - \gamma(t)) - f_r(\gamma(s) - \gamma(0)) \\ &\ge |\gamma(r) - \gamma(0)| - |\gamma(r) - \gamma(t)| - |\gamma(s) - \gamma(0)| \\ &\ge r - b - (r - t) - s = t - s - b. \end{aligned}$$

We conclude that $f(\gamma(t) - \gamma(s)) \ge t - s - b$ for any $t \ge s \ge 0$ and that $f \in S(X^*)$. The result follows.

THEOREM 3.2. *A convex subset $C \subset X$ has the AFPP if and only if for every sequence $\{x_n\}_{n=1}^\infty \subset C$ such that $|x_n| \to \infty$ as $n \to \infty$ and every $f \in S(X^*)$,*

$$\limsup_{n \to \infty} f(x_n/|x_n|) < 1.$$

PROOF. If C does not have the AFPP, then by Theorem 2.4 it contains a directional curve $\gamma(t)$. Taking $x_n = \gamma(n)$ and the functional $f \in S(X^*)$ given by Lemma 3.1 we certainly have $\lim_{n \to \infty} f(x_n/|x_n|) = 1$.

The proof of necessity is similar to the proof of sufficiency of Theorem 2.4. Suppose we have an unbounded sequence $\{x_n\}_{n=1}^\infty \in C$ and a functional $f \in S(X^*)$ such that $\lim_{n \to \infty} f(x_n/|x_n|) = 1$. We may assume that $f(x_n/|x_n|) \ge 1 - 1/2^n$ for all n. We now define inductively sequences $\{n_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty \subset C$ such that $\{y_i\}$ is a directional sequence. We set $n_1 = 1$, $y_1 = x_1$ and for $i \ge 1$ we choose n_{i+1} such that

$$|x_{n_{i+1}}| \ge 2|x_{n_i}| \quad \text{and} \quad |x_{n_{i+1}} - y_i|/|x_{n_{i+1}}| \le 1 + 1/2^{i+1}$$

and set

$$y_{i+1} \equiv \left(\sum_{j=1}^{i+1} x_{n_j} / |x_{n_j}| \right) \Big/ \sum_{j=1}^{i+1} 1/|x_{n_j}| .$$

A computation similar to the one given in the proof of Theorem 2.4 shows that $\{y_i\}_{i=1}^{\infty}$ is a directional sequence, hence C does not have the AFPP.

REMARK 3.3. In Theorem 3.2 we may replace "$f \in S(X^*)$" by "f which is an extreme point of $S(X^*)$". Indeed, if C contains a directional curve $\gamma(t)$ then

$$a = \inf_{g \in S(X^*)} \sup_{t > 0} \left(t - g(\gamma(t) - \gamma(0)) \right)$$

is finite by Lemma 3.1. Note that the "sup" can be replaced by "lim" since the function $H(t) = t - g(\gamma(t) - \gamma(0))$ is nondecreasing for $t > 0$. The infimum a is attained since if the sequence

$$a_n \equiv \sup_{t > 0} \left(t - g_n(\gamma(t) - \gamma(0)) \right)$$

converges to a, then for any $t > 0$, $t - g(\gamma(t) - \gamma(0)) \leq a$ where g is a w^* accumulation point of the net $\{g_n\}$. The set

$$F \equiv \left\{ f \in S(X^*); \quad \sup_{t > 0} \left(t - f(\gamma(t) - \gamma(0)) \right) = a \right\}$$

is a nonempty w^*-compact and convex subset of $S(X^*)$. Hence by the Krein-Milman theorem it contains an extreme point. Since F is clearly an extremal subset of $S(X^*)$, i.e. $f, g \in S(X^*)$ and $(f+g)/2 \in F \Rightarrow f, g \in F$, this extreme point is also an extreme point of $S(X^*)$.

EXAMPLE 3.4. Consider the subset C of c_0 given by

$$C \equiv \left\{ \{x_i\}_{i=1}^{\infty} \in c_0; \ |x_i| \leq a_i \text{ for every } i \right\}$$

where $\{a_n\}$ is a positive unbounded sequence. C is clearly a closed unbounded convex subset of c_0. We claim that C has the AFPP. Indeed $c_0^* = l_1$, the extreme points of l_1 are $\{\pm e_n\}_{n=1}^{\infty}$ where $\{e_n\}$ denote the coordinate functionals, all of them are bounded on C, so for any such functional f and any sequence $\{y_n\}_{n=1}^{\infty} \subset C$ for which $|y_n| \to \infty$, $f(y_n)/|y_n| \to 0$. The result follows from Remark 3.3.

The next result shows that we can replace directionally bounded by linearly bounded in Theorem 2.4 if and only if the Banach space X is reflexive.

Proposition 3.5. *In a Banach space X, every closed and convex subset that is linearly bounded is directionally bounded if and only if X is reflexive.*

Proof. The sufficiency part follows from Reich's result [7] and Theorem 2.4, but we give a direct proof for completeness. If X is reflexive and $C \subset X$ is a closed and convex subset which is not directionally bounded, then there is a directional curve $\gamma(t)$ contained in C. Let $t_n \uparrow \infty$ be a sequence such that

$$\gamma(t_n)/|\gamma(t_n)| \xrightarrow{w} v.$$

By Lemma 3.1 there is $f \in S(X^*)$ such that $f(\gamma(t_n))/|\gamma(t_n)| \to 1$, hence $f(v) = 1$ and $v \in S(X)$. Fix any $y \in C$. We claim that the half line $\{y + sv \mid s > 0\}$ is contained in C. Indeed, for any $s > 0$,

$$\text{weak} \lim_{n\to\infty} (|\gamma(t_n)| - s)y/|\gamma(t_n)| + s\gamma(t_n)/|\gamma(t_n)| = y + sv \in C.$$

Conversely, suppose X is not reflexive. Then by James' theorem there is a functional $f \in S(X^*)$ which does not attain its maximum on $S(X)$. We choose a sequence $\{y_i\}$ from $S(X)$ such that

$$\sum_{i=1}^{\infty} (1 - f(y_i)) \le 1.$$

We define

$$x_n = \sum_{i=1}^{n} y_i \qquad \text{for } n \ge 1,$$

and set $C = \text{clco}\{x_n\,;\, n \ge 1\}$. C is not directionally bounded since it contains the directional sequence $\{x_n\}_{n=1}^{\infty}$:

$$\sum_{i=1}^{n-1} |x_i - x_{i+1}| - |x_1 - x_n| = \sum_{i=2}^{n} |y_i| - \left|\sum_{i=2}^{n} y_i\right| \le n - 1 - \sum_{i=2}^{n} f(y_i) \le 1.$$

We claim that C is linearly bounded. For

$$z = \sum_{i=1}^{n} c_i x_i, \qquad \text{where } c_i \ge 0, \quad \text{for all } i \text{ and } \sum_{i=1}^{n} c_i = 1,$$

we have

$$\sum_{i=1}^{n} ic_i \ge f(z) \ge \sum_{i=1}^{n} c_i(i-1) = \sum_{i=1}^{n} ic_i - 1$$

so

$$f(z)/|z| \geq \left(\sum_{i=1}^{n} ic_i - 1\right)\bigg/\sum_{i=1}^{n} c_i|x_i| \geq 1 - 1\bigg/\sum_{i=1}^{n} ic_i.$$

It follows that if C contains a half line $\{y + sv | s > 0\}$ then

$$\lim_{s\to\infty} f(y + sv)/|y + sv| = 1,$$

so $f(v) = 1$, contrary to our assumption.

Next we shall deal with construction of unbounded convex subsets which have the AFPP. We shall use the following definition:

Definition 3.6. A sequence $\{x_n\}_{n=1}^{\infty} \subset S(X)$ is called a (P)-sequence if for every functional $f \in S(X^*)$ there is a functional $g \in S(X^*)$ such that

$$\limsup_{n\to\infty} f(x_n) < \liminf_{n\to\infty} g(x_n).$$

Lemma 3.7. *If X contains a* (P)*-sequence then X contains an unbounded convex subset C which is directionally bounded.*

Proof. Let $\{x_n ; n \geq 1\} \subset S(X)$ be a (P)-sequence, define $y_n = nx_n$ and consider $C = \mathrm{co}\{y_n ; n \geq 1\}$. We claim that C is directionally bounded. If not, there is $f \in S(X^*)$ and $\{z_n\}_{n=1}^{\infty}$ in C such that $|z_n| \to \infty$ and $\lim_{n\to\infty} f(z_n)/|z_n| = 1$. But since $\{x_n\}$ is a (P)-sequence, there is $g \in S(X^*)$ such that $g(x_n) > (1 + 3\epsilon)f(x_n)$ for $n \geq n_0$ and some $\epsilon > 0$. This implies that $g(z_n) > (1 + 2\epsilon)f(z_n)$ for $n \geq n_1$. We get for $n \geq n_2$,

$$1 + \epsilon < g(z_n)/|z_n| \leq 1,$$

a contradiction.

Proposition 3.8. *If X does not contain an isomorphic copy of l_1, then there is a convex unbounded subset C of X which has the AFPP.*

Proof. It is clear that we may assume that X is separable. First, we consider the case when X is reflexive. By Proposition 3.5, it is sufficient to find an unbounded closed and convex subset that is linearly bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence in $S(X)$ and for any $n \geq 1$ choose $f_n \in J(x_n)$. Note that for any $y \in X$, $|y| = \sup\{f_n(y) | n \geq 1\}$. Next we define a sequence $\{y_n\}_{n=1}^{\infty}$ as follows:

$$y_n \in \bigcap_{i=1}^{n} \ker f_i \quad \text{and} \quad |y_n| = n.$$

We set $C = \mathrm{clco}\{y_n\}_{n=1}^{\infty}$ and claim that C is linearly bounded. Indeed, if for some v such that $|v| = 1$, $\{y + sv | s > 0\} \subset C$, then there exists some n_0 such that

$$f_{n_0}(v) > 0,$$

hence

$$f_{n_0}(y + nv) \to \infty \qquad \text{as } n \to \infty.$$

But by our construction,

$$\sup_{x \in C} f_{n_0}(x) \leq n_0 - 1,$$

a contradiction.

Now, consider the case when X is not reflexive. Since X does not contain l_1 isomorphically, the Odel–Rosenthal theorem (see [2]) states that $S(X)$ is w^*-sequentially dense in $S(X^{**})$. Let $\phi \in S(X^{**})$ be such that the maximum of ϕ on $S(X^*)$ is not attained. Choose $\{x_n\}_{n=1}^{\infty} \subset S(X)$ such that $x_n \xrightarrow{w^*} \phi$ as $n \to \infty$ in X^{**}. Clearly $\{x_n\}$ is a (P)-sequence and the result follows by Lemma 3.7.

It is interesting to note that although the above proposition does not cover the case $X = l_1$, its conclusion is still valid in this case, too.

EXAMPLE 3.9. We shall show that there is an unbounded convex subset of l_1 which has the AFPP. By Lemma 3.7 it is enough to find a (P)-sequence in l_1. Let $\{e_n\}_{n=1}^{\infty}$ denote the standard base of l_1 and let $\alpha = (\alpha_n)_{n=1}^{\infty} \subset l_1$ be such that

$$\alpha_i > 0 \qquad \text{for all } i \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1.$$

Consider the sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n = \alpha - e_n$. For $m \geq 1$ let $f_m = (a_n^{(m)})_{n=1}^{\infty} \in S(l_\infty)$ be defined by $a_n^{(m)} = 1$ for $n \leq m$ and $a_n^{(m)} = -1$ for $n > m$. We have

$$f_m(x_n) = 1 + \sum_{i=1}^{m} \alpha_i - \sum_{i=m+1}^{\infty} \alpha_i$$

for $n > m$. Hence

$$\sup_{m \geq 1} \lim_{n \to \infty} f_m(x_n) = 2 = \lim_{n \to \infty} |x_n|.$$

We claim that for every $f \in S(l_\infty)$, $\limsup f(x_n) < 2$. For $f = (a_n)_{n=1}^{\infty} \in S(l_\infty)$ we have

$$f(x_n) = f(\alpha) - a_n \qquad \text{for all } n.$$

If $a_n > 0$ for all n then $f(x_n) \leq 1$ for all n. Otherwise, if $a_l \leq 0$ for some l then $f(\alpha) \leq 1 - \alpha_l$ and so $\forall n$, $f(x_n) \leq 2 - \alpha_l$. In any case $\limsup f(x_n) < 2$, hence $\{x_n/|x_n|\}_{n=1}^{\infty}$ is a (P)-sequence.

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